Exercise X, Theory of Computation 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long.

These problems are taken from various sources at EPFL and on the Internet, too numerous to cite individually.

1 A formula is in Disjunctive Normal Form (DNF) if it is an OR ($\vee$) of a number of terms, where each term is an AND ($\wedge$) of literals, as for example $(x \wedge y \wedge \bar{z}) \vee (\bar{y} \wedge z) \vee (\bar{x} \wedge \bar{y})$. Define

$$\text{DNF-SAT} = \{\langle \varphi \rangle : \varphi \text{ is a DNF formula and } \varphi \text{ is satisfiable}\}.$$

1a Prove that DNF-SAT is in **P**.

1b What is wrong with the following reasoning?

Suppose we are given a 3CNF formula, and we want to know if it is satisfiable. We can repeatedly use the distributive law

$$(\varphi_1 \vee \varphi_2) \wedge \psi \equiv (\varphi_1 \wedge \psi) \vee (\varphi_2 \wedge \psi)$$

to construct an equivalent DNF formula. For instance,

$$(x \vee y \vee \bar{z}) \wedge (\bar{x} \vee \bar{y}) \equiv (x \wedge \bar{x}) \vee (x \wedge \bar{y}) \vee (y \wedge \bar{x}) \vee (y \wedge \bar{y}) \vee (\bar{z} \wedge \bar{x}) \vee (\bar{z} \wedge \bar{y}).$$

Then, using the algorithm from Part (a) we can determine, in polynomial time, whether the resulting DNF formula is satisfiable. We have just solved 3SAT in polynomial time. Since 3SAT is **NP**-complete, we conclude that $\mathbf{P} = \mathbf{NP}$.

2 Prove that the following language is **NP**-complete:

$$L = \{\langle G = (V, E), k \rangle : \text{there exist disjoint } S_1, S_2 \subseteq V, \text{ such that } |S_1|, |S_2| \geq k, \\ S_1 \text{ is an independent set in } G \text{ and } S_2 \text{ is a clique in } G\}.$$

3 Recall that a *matching* in an undirected graph $G = (V, E)$ is a subset $M \subseteq E$ such that no two edges in M share an endpoint. We say that a matching M is *well separated* if for any distinct pair of edges $e, e' \in M$ each endpoint of e is at distance at least 2 from each endpoint of e'. Prove that the following problem is **NP**-complete:

$$\textsc{SepMatch} = \{\langle G, k \rangle : G \text{ contains a well separated matching of size } k\}$$

4 Prove that the following problem is **NP**-hard: Given n integers $a_1, a_2, \ldots, a_n$, is there a set $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i?$$